Combinatorial Counting - 3.1 Functions and subsets

Notation

$[n] = \{1, 2, \ldots, n\}$

$[a; b] = \{a, a + 1, \ldots, b\}$

If $a > b$, then $[a; b] = \emptyset$.

Problem: Count the number of mappings $f : [n] \to [m]$, where $m, n \geq 1$.

1: Count the number of mappings $f : [7] \to [4]$.
Practical version: Each of 7 students can pick one of 4 flavors of ice cream (unlimited supply of each flavor). How many different configurations are there?

Solution: I need to map each $1, 2, \ldots, 7$ to a number in $[4]$. There are 4 ways to do this for each 1,2,...,7 so the answer is 4^7. If you have a combinatorial problem and do not know how to start, start with a small example. On a small example, one can sometimes observe a pattern and/or develop a method, that generalizes.

2: Show that the number of mappings $f : [n] \to [m]$, where $m, n \geq 1$ is exactly m^n using induction on n.

Solution: For $n = 1$, the here are $m = m^1$ choices. For $[n + 1]$, there are m^n choices for $[n]$ and m choices for mapping of $n + 1$, which gives $m^n \cdot m = m^{n+1}$.

3: Let alphabet A has m letters. Show that there are m^n words of length n over the alphabet A.

Solution: Every word corresponds to a mapping $f : [n] \to [m]$. And the count is m^n.

4: What should be $f : \emptyset \to [m]$?

Solution: This corresponds to $f : [0] \to [m]$. The previous result would suggest to count it as $m^0 = 1$. Let's be more rigorous. What is $f : X \to Y$ for some X and Y? It looks like

$$f \subseteq \{(x, y) : x \in X, y \in Y\},$$

where every $x \in X$ is in exactly one pair. If $X = \emptyset$, then $f = \emptyset$ works. Hence there is indeed one mapping from $\emptyset \to Y$. Checking the boundary cases allows to skip some exceptions later.

Claim: Every n-element set X has 2^n distinct subsets, including the empty set and itself.

$|\mathcal{P}(X)| = 2^{|X|}$, where $\mathcal{P}(X)$ denotes all subsets of X.

5: Prove the claim by induction on n.

Solution: For $n = 0$, we have 1 subset, the empty set. And $2^0 = 1$. For induction step, let $|X| = n + 1$ and $y \in X$ be any element in X. Consider $Y = X \setminus \{y\}$. Then $|Y| = n$ and $|\mathcal{P}(Y)| = 2^{|Y|} = 2^n$. Now for every set $Z \in \mathcal{P}(Y)$, we can create two sets in $\mathcal{P}(Y)$ by considering Z and $Z \cup \{y\}$. This gives $2 \cdot 2^n = 2^{n+1}$ subsets. Idea - every subset of X either contains y or does not contain y.

6: Prove the claim considering the characteristic function.

Solution: Every subset A of X is associated with a function $f : X \to [0,1]$ such that $f(x) = 1$ iff $x \in A$. There are 2^n such functions as we observed earlier.

A mapping $f : X \to Y$ is injective, if for every $a, b \in X$, if $a \neq b$, then $f(a) \neq f(b)$.

7: Let there be 7 students, and 20 different candies. Each candy is unique and only once. How many ways can students pick candies if each student get one candy? *What about the other candies?*

Solution: $20 \cdot 19 \cdot 18 \cdots 14 = 390,700,800$

8: Show that the numbers $m, n \geq 0$, there exist

$$\prod_{i=0}^{n-1}(m-i)$$

injective mappings of an n-element set to an m-element set. What happens if $m < n$ or either of them is 0?

Solution: First element in $[n]$ has m choices, then $m-1$ choices and so on. Notice that if $n > m$, then the product contains 0. If $n = 0$, then the product is empty and we say that empty product is $= 1$. If $m = 0$, then it contains $m - 0 = 0$ entry and the product is 0 if $n > 0$.

9: What is the probability that random 8 people have the same birthday? (ignore Feb 29) Try at home - how about with 20 people?

Solution: The probability is

$$\frac{\text{number of good events}}{\text{number of all events}} = 1 - \frac{\text{number of bad events}}{\text{number of all events}} = 1 - \frac{\prod_{i=0}^{8-1} 365 - i}{365^8} \approx 0.075$$

The chance is about 7.5%.